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# Buckling of Transversely Isotropic Mindlin Plates

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**General equations are derived that describe the motion of a transversely isotropic Mindlin plate subjected to initial stress and displacement. These equations can be used to investigate static deflections, free and forced vibrations, wave propagation etc., but for present purposes are specialized to investigate buckling behavior. It is shown that transverse isotropy, which accentuates the effects of shear deformation, induces decreases in the buckling loads, these decreases becoming larger as boundary restraint is increased. Hence, as the plate becomes more transversely isotropic, the addition of boundary restraint is progressively less effective in raising the buckling loads of the transversely isotropic plate under consideration.**

## Introduction

RECENT articles have discussed the increased importance of shear deformation when treating a variety of transversely isotropic beam and plate problems.[1–4] In particular, Brunelle[4] has demonstrated the deleterious effects of transverse shear on the buckling loads of transversely isotropic beams and has pointed out that increased boundary restraint produces much less of an increase in buckling loads for transversely isotropic beams than for their counterpart isotropic beams. This suggests, and it is born out in the sequel, that the effects of transverse shear may also be an important factor in analyzing the stability of transversely isotropic Mindlin plates. To this end the present paper derives modified Mindlin† plate equations that include the effects of transverse isotropy, initial stress, and initial displacement. These plate equations are suitable for investigating static deflections, free and forced vibrations, wave propagation, etc., as well as elastic stability. For present purposes however these equations are then specialized so that only elastic stability problems may be treated. In particular the elastic stability of a rectangular plate, with two parallel sides simply supported and the remaining two sides subjected to a variety of boundary conditions, is treated in detail.

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† The definition of a Mindlin plate is a plate which possesses rotary inertia and shear deformation characteristics as described in Ref. 5.

## Plate Equations

Following Mindlin,[5] the $x$, $y$, $z$ displacements, respectively, are assumed of the form

$$u(x,y,z,t) \cong z\psi_x(x,y,t) \tag{1a}$$

$$v(x,y,z,t) \cong z\psi_y(x,y,t) \tag{1b}$$

$$w(x,y,z,t) \cong \tilde{w}(x,y,t) = w_0(x,y) + w(x,y,t)‡ \tag{1c}$$

where $\psi_x$ and $\psi_y$ are rotations, $\tilde{w}$ is the total approximate transverse deflection, $w_0$ is the initial stress-free transverse deflection, and $w(x,y,t)$ is the elastic transverse deflection. Hence the approximate strain field is given by

$$\epsilon_x = \frac{\partial u}{\partial x} \cong z\frac{\partial \psi_x}{\partial x}, \quad \epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \cong \frac{1}{2}\left(\psi_x + \frac{\partial w}{\partial x}\right) \tag{2a}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \cong z\frac{\partial \psi_y}{\partial y}, \quad \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \cong \frac{1}{2}\left(\psi_y + \frac{\partial w}{\partial y}\right) \tag{2b}$$

‡ It is to be noted that the definition of $w$ has been modified to include initial displacement.

$$\epsilon_z = \frac{\partial w}{\partial z} \cong 0, \quad \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \cong \frac{z}{2}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) \tag{2c}$$

Assuming $\sigma_z \ll \sigma_x$ and $\sigma_y$ and introducing elastic constants of the form $E_x = E_y = E$, $E_z \to \infty$, $G_{xy} = G$, and $G_{xz} = G_{yz} = \kappa^2 G^*$ (where $\kappa^2$ is Mindlin's shear correction factor which we assume to be $\pi^2/12$) the approximate stress-strain laws, for a transversely isotropic material, become

$$\sigma_x \cong [E/(1-\nu^2)](\epsilon_x + \nu\epsilon_y),$$
$$\sigma_y \cong [E/(1-\nu^2)](\epsilon_y + \nu\epsilon_x)$$
$$\sigma_{xy} \cong 2G\epsilon_{xy}, \quad \sigma_{xz} \cong 2\kappa^2 G^*\epsilon_{xz}, \quad \sigma_{yz} \cong 2\kappa^2 G^*\epsilon_{yz} \tag{3}$$

Using the results of Eq. (2), Eq. (3) becomes,

$$\sigma_x \cong \frac{Ez}{1-\nu^2}\left(\frac{\partial \psi_x}{\partial x} + \nu\frac{\partial \psi_y}{\partial y}\right), \quad \sigma_y \cong \frac{Ez}{1-\nu^2}\left(\frac{\partial \psi_y}{\partial y} + \nu\frac{\partial \psi_x}{\partial x}\right)$$
$$\sigma_{xy} \cong Gz\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right), \quad \sigma_{xz} \cong \kappa^2 G^*\left(\psi_x + \frac{\partial w}{\partial x}\right), \tag{4}$$
$$\sigma_{yz} \cong \kappa^2 G^*\left(\psi_y + \frac{\partial w}{\partial y}\right)$$

Now introducing the following moment and stress resultants

$$M_x = -\int_{-h/2}^{h/2} \sigma_x z dz, \quad M_{xy} = -\int_{-h/2}^{h/2} \sigma_{xy} z dz,$$
$$M_y = -\int_{-h/2}^{h/2} \sigma_y z dz \tag{5}$$
$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz} dz, \quad Q_y = \int_{-h/2}^{h/2} \sigma_{yz} dz$$

and using Eq. (4), Eq. (5) becomes

$$M_x = -\mathfrak{D}\left(\frac{\partial \psi_x}{\partial x} + \nu\frac{\partial \psi_y}{\partial y}\right), M_y = -\mathfrak{D}\left(\frac{\partial \psi_y}{\partial y} + \nu\frac{\partial \psi_x}{\partial x}\right)$$
$$M_{xy} = -[(1-\nu)/2]\mathfrak{D}(\partial\psi_x/\partial y + \partial\psi_y/\partial x) \tag{6}$$
$$Q_x = \kappa^2 G^* h(\psi_x + \partial w/\partial x), \quad Q_y = \kappa^2 G^* h(\psi_y + \partial w/\partial y)$$

where $\mathfrak{D} = Eh^3/12(1-\nu^2)$.

Equilibrating the forces and moments on a *deformed* plate element of constant thickness $h$ yields three equations of motion[5,6]

$$q + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x\frac{\partial^2(w+w_0)}{\partial x^2} + 2N_{xy}\frac{\partial^2(w+w_0)}{\partial x \partial y} + N_y\frac{\partial^2(w+w_0)}{\partial y^2} - \rho h\ddot{w} = 0 \tag{7}$$

$$\partial M_x/\partial x + \partial M_{yx}/\partial y + Q_x + \ddot{\psi}_x \rho h^3/12 = 0 \tag{8}$$

$$\partial M_y/\partial y + \partial M_{xy}/\partial x + Q_y + \ddot{\psi}_y \rho h^3/12 = 0 \tag{9}$$

where $q$ = distributed transverse loading; $N_x$, $N_y$, $N_{xy}$ = initial in-plane resultant due to initial stress; $\rho$ = mass density of plate; $h$ = plate thickness.

Notice that if $w_0 = N_x = N_{xy} = N_y = 0$ (zero initial displacement and stress), Eqs. (7–9) are exactly Mindlin's plate equations.[5]

Using the results of (6), and defining

$$S_x = -N_x/\mathfrak{D}, \quad S_{xy} = -N_{xy}/\mathfrak{D}, \quad S_y = -N_y/\mathfrak{D}$$
$$M = \rho h^3/12\mathfrak{D}, \quad M^* = \rho h/\mathfrak{D}, \quad Q = \kappa^2 G^* h/\mathfrak{D}$$

Equations (7–9) are finally given as

$$-Q\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + S_x\frac{\partial^2(w+w_0)}{\partial x^2} + 2S_{xy}\frac{\partial^2(w+w_0)}{\partial x\partial y} + S_y\frac{\partial^2(w+w_0)}{\partial y^2} + M^*\ddot{w} = \frac{q}{\mathfrak{D}} \tag{10}$$

$$\frac{\partial^2\psi_x}{\partial x^2} + \frac{1}{2}(1-\nu)\frac{\partial^2\psi_x}{\partial y^2} + \frac{1}{2}(1+\nu)\frac{\partial^2\psi_y}{\partial x\partial y} - Q\left(\psi_x + \frac{\partial w}{\partial x}\right) - \ddot{\psi}_x M = 0 \tag{11}$$

$$\frac{\partial^2\psi_y}{\partial y^2} + \frac{1}{2}(1-\nu)\frac{\partial^2\psi_y}{\partial x^2} + \frac{1}{2}(1+\nu)\frac{\partial^2\psi_x}{\partial x\partial y} - Q\left(\psi_y + \frac{\partial w}{\partial y}\right) - \ddot{\psi}_y M = 0 \tag{12}$$

Equations (10–12) are similar to a linearized version of those derived by Herrman and Armenakas[7] for isotropic elastic plates under initial stress and displacement. Expressions alternate to Eqs. (10–12) as well as an uncoupled equation for the transverse deflection $w$ are presented in Appendix A.

It bears repeating that Eqs. (10–12) are suitable for analyzing the whole spectrum of statics and dynamics problems that can be associated with a transversely isotropic Mindlin plate under initial stress and displacement. However, these equations are now specialized to describe the elastic stability of a plate simply supported on the two sides $x = 0,a$.

## Homogeneous Stability Equations of a Plate Simply Supported at $x = 0$, $a$ with $S_x$ and $S_y$ Acting

Letting $q = S_{xy} = w_0 = \partial^2/\partial t^2 = 0$, introducing the differential operators $D^n = d^n/dy^n$, and assuming the displacement and rotations to be given by

$$w(x,y) = W(y)\sin(m\pi x/a)$$
$$\psi_x(x,y) = \Psi_x(y)\cos(m\pi x/a) \tag{13}$$
$$\psi_y(x,y) = \Psi_y(y)\sin(m\pi x/a)$$

where $m$ is the number of half-waves in the $x$ direction, it is seen that simply supported boundary conditions on the sides $x = 0$, $a$ are satisfied, and that the operator equation equivalents of Eqs. (10–12) become

$$[A_{ij}]\begin{Bmatrix} W \\ \Psi_x \\ \Psi_y \end{Bmatrix} = \{0\} \tag{14}$$

where the elements of $[A_{ij}]$ are given by

$$A_{11} = (1 - S_y/Q)D^2 - (m\pi/a)^2(1 - S_x/Q),$$
$$A_{12} = -m\pi/a, \quad A_{13} = D$$
$$A_{21} = -Q(m\pi/a), \quad A_{22} = \tfrac{1}{2}(1-\nu)D^2 - [Q + (m\pi/a)^2]$$
$$A_{23} = \tfrac{1}{2}(1+\nu)(m\pi/a)D, \quad A_{31} = -QD$$
$$A_{32} = -\tfrac{1}{2}(1+\nu)(m\pi/a)D,$$
$$A_{33} = D^2 - [Q + \tfrac{1}{2}(1-\nu)(m\pi/a)^2]$$

The uncoupled operator equation for $W$ (and for $\Psi_x$ and $\Psi_y$ also) is given by the determinant of $[A_{ij}]$ and may be expressed in the following factored form

$$(A_4 D^4 + A_2 D^2 + A_0)(C_2 D^2 + C_0)W = 0 \tag{15}$$

where

$$A_0 = -(m\pi/a)^2[(m\pi/a)^2(S_x/Q - 1) + S_x] \tag{16a}$$

$$A_2 = (m\pi/a)^2(S_x/Q - 1) + (m\pi/a)^2(S_y/Q - 1) + S_y \tag{16b}$$

$$A_4 = (1 - S_y/Q), \quad C_0 = -\{Q + [(1-\nu)/2](m\pi/a)^2\}$$
$$C_2 = (1-\nu)/2 \tag{16c}$$

Corresponding to Eq. (13), the moment and shear resultants

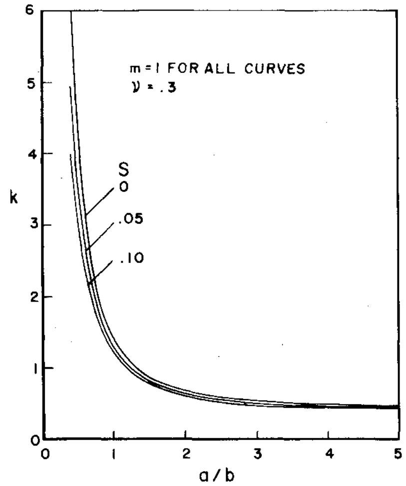


Fig. 1 Buckling coefficient for $x$ sides simply supported, one side free and one side simply supported.

needed for specifying boundary conditions on the $y$ faces are given by

$$M_y = [\nu(m\pi/a)\Psi_x - \Psi_y']\mathfrak{D}\sin(m\pi x/a) \quad (17a)$$

$$M_{yx} = -[(1-\nu)/2][\Psi_x' + (m\pi/a)\Psi_y]\mathfrak{D}\cos(m\pi x/a) \quad (17b)$$

$$Q_y = [\Psi_y + W']\kappa^2 G^* h \sin(m\pi x/a) \quad (17c)$$

A list of common boundary conditions is given in Appendix B.

The roots of the biquadratic factor in Eq. (15) are $D = \pm\alpha$, $\pm i\beta$ and the roots of the quadratic factor in Eq. (15) are $D = \pm\gamma$, where

$$\alpha^2 = -A_2/2A_4 + [(A_2/2A_4)^2 - (A_0/A_4)]^{1/2} \quad (18a)$$

$$\beta^2 = A_2/2A_4 + [(A_2/2A_4)^2 - (A_0/A_4)]^{1/2} \quad (18b)$$

$$\gamma^2 = 2Q/(1-\nu) + (m\pi/a)^2 \quad (18c)$$

Hence the $W$, $\Psi_x$, and $\Psi_y$ solutions of Eq. (15) which satisfy any two of the parent equations, say Eqs. (11) and (12), are given by§

$$W(y) = A\sinh\alpha y + B\cosh\alpha y + C\sin\beta y + D\cos\beta y \quad (19)$$

$$\Psi_x(y) = (m\pi/a)[\delta_1 A\sinh\alpha y + \delta_1 B\cosh\alpha y - \delta_2 C\sin\beta y - \delta_2 D\cos\beta y] + E\sinh\gamma y + F\cosh\gamma y \quad (20)$$

$$\Psi_y(y) = \delta_1\alpha A\cosh\alpha y + \delta_1\alpha B\sinh\alpha y - \delta_2\beta C\cos\beta y + \delta_2\beta D\sin\beta y + (1/\gamma)(m\pi/a)E\cosh\gamma y + (1/\gamma)(m\pi/a)F\sinh\gamma y \quad (21)$$

where

$$\delta_1 = (Q/\Delta_1)\{\tfrac{1}{2}(1-\nu)[\alpha^2 - (m\pi/a)^2] - Q\}$$

$$\delta_2 = (Q/\Delta_2)\{\tfrac{1}{2}(1-\nu)[\beta^2 + (m\pi/a)^2] + Q\}$$

$$\Delta_1 = [\tfrac{1}{2}(1-\nu)\alpha^2 - Q - (m\pi/a)^2][\alpha^2 - Q - \tfrac{1}{2}(1-\nu)(m\pi/a)^2] + \tfrac{1}{4}(1+\nu)^2(m\pi/a)^2\alpha^2$$

$$\Delta_2 = [\tfrac{1}{2}(1-\nu)\beta^2 + Q + (m\pi/a)^2][\beta^2 + Q + \tfrac{1}{2}(1-\nu)(m\pi/a)^2] - \tfrac{1}{4}(1+\nu)^2(m\pi/a)^2\beta^2$$

In particular note the degenerate form of Eq. (19) which does not contain $\sinh\gamma y$ and $\cosh\gamma y$ terms. Equations (19–21) may be used, along with appropriate boundary conditions at $y = 0, b$, to solve buckling problems with biaxial compressions $S_x$ and $S_y$ acting. In what follows a more modest class of problems are solved; those with uniaxial compression (i.e., $S_y = 0$).

§ From this point on, $D$ is a constant of integration, not an operator.

## Buckling of Plates Simply Supported at $x = 0$, $a$ with a Uniaxial Compression $S_x$ Acting

With $S_y = 0$, Eq. (18) reduces to the following set of relations,

$$\alpha^2 = \left(\frac{m\pi}{a}\right)^2\left(1 - \frac{S_x}{2Q}\right) + \frac{m\pi}{a}\left[\left(\frac{m\pi}{a}\right)^2\left(\frac{S_x}{2Q}\right)^2 + S_x\right]^{1/2} \quad (22a)$$

$$\beta^2 = \left(\frac{m\pi}{a}\right)^2\left(\frac{S_x}{2Q} - 1\right) + \frac{m\pi}{a}\left[\left(\frac{m\pi}{a}\right)^2\left(\frac{S_x}{2Q}\right)^2 + S_x\right]^{1/2} \quad (22b)$$

$$\gamma^2 = 2Q/(1-\nu) + (m\pi/a)^2 \quad (22c)$$

and for convenience two new parameters $S$ and $k$ are defined as

$$S = (1/Q)(\pi/b)^2 \equiv (E/G^*)(h/b)^2/(1-\nu^2)$$

$$k = (b/\pi)^2 S_x$$

Note that $k$ is the usual buckling coefficient and that $S$ is the only parameter that includes transverse isotropy, that is the $E/G^*$ ratio. With these results in mind, a series of four problems and their results are now presented.

### Plate Simply Supported at $y = 0$, Free at $y = b$

Applying the boundary conditions

$$W(0) = \Psi_x(0) = \Psi_y'(0) = 0, \quad W'(b) + \Psi_y(b) = 0$$

$$\Psi_x'(b) + (m\pi/a)\Psi_y(b) = 0, \quad \nu(m\pi/a)\Psi_x(b) - \Psi_y'(b) = 0$$

leads to the conclusions that $B = D = F = 0$ and that

$$[a_{ij}]\begin{Bmatrix} A \\ C \\ E \end{Bmatrix} = 0 \quad (23)$$

where

$$a_{11} = (1+\delta_1)\alpha/\cosh\gamma b$$

$$a_{12} = (1-\delta_2)\beta\cos\beta b/\cosh\alpha b\cosh\gamma b$$

$$a_{13} = (1/\gamma)(m\pi/a)/\cosh\alpha b$$

$$a_{21} = 2\delta_1\alpha/\cosh\gamma b$$

$$a_{22} = -2\delta_2\beta\cos\beta b/\cosh\alpha b\cosh\gamma b$$

$$a_{23} = [\gamma(a/m\pi) + (1/\gamma)(m\pi/a)/\cosh\alpha b$$

$$a_{31} = [\nu(m\pi/a)^2 - \alpha^2]\delta_1\tanh\alpha b/\cosh\gamma b$$

$$a_{32} = -[\nu(m\pi/a)^2 + \beta^2]\delta_2\sin\beta b/\cosh\alpha b\cosh\gamma b$$

$$a_{33} = (\nu - 1)(m\pi/a)\tanh\gamma b/\cosh\alpha b$$

The stability determinant $|a_{ij}| = 0$ yields values of $k$ for given values of $a/b$, $S$, $m$, and $\nu$; the minimum $k$ values¶ for $\nu = 0.3$ are shown in Fig. 1. The mode shape is then determined by using (23).

### Plate Clamped at $y = 0$, Free at $y = b$

Applying the boundary conditions

$$W(0) = \Psi_x(0) = \Psi_y(0) = 0, \quad \nu(m\pi/a)\Psi_x(b) - \Psi_y'(b) = 0$$

$$(a/m\pi)\Psi_x'(b) + \Psi_y(b) = 0, \quad W'(b) + \Psi_y(b) = 0$$

¶ That is, the lower envelope of the $k$ values.

leads to the conclusions that

$$D = -B, \quad E = \gamma(a/m\pi)[\delta_2\beta C - \delta_1\alpha A],$$
$$F = -(m\pi/a)(\delta_1 + \delta_2)B \tag{24}$$

and that

$$[a_{ij}]\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\} \tag{25}$$

where**

$$a_{11} = \alpha[(1 + \delta_1)\cosh\alpha b - \delta_1 \cosh\gamma b]$$

$$a_{12} = \alpha(1 + \delta_1)\sinh\alpha b + \beta(1 - \delta_2)\sin\beta b - (1/\gamma)(m\pi/a)^2(\delta_1 + \delta_2)\sinh\gamma b$$

$$a_{13} = \beta[(1 - \delta_2)\cos\beta b + \delta_2\cosh\gamma b]$$

$$a_{21} = \delta_1[\{(m\pi/a)^2\nu - \alpha^2\}\sinh\alpha b + \alpha\gamma(1 - \nu)\sinh\gamma b]$$

$$a_{22} = \delta_1\{(m\pi/a)^2\nu - \alpha^2\}\cosh\alpha b + \delta_2\{(m\pi/a)^2\nu + \beta^2\}\cos\beta b + (m\pi/a)^2(\delta_1 + \delta_2)(1 - \nu)\cosh\gamma b$$

$$a_{23} = \delta_2[\gamma\beta(\nu - 1)\sinh\gamma b - \{(m\pi/a)^2\nu + \beta^2\}\sin\beta b]$$

$$a_{31} = \delta_1\alpha[2\cosh\alpha b - \{1 + \gamma^2(a/m\pi)^2\}\cosh\gamma b]$$

$$a_{32} = 2\delta_1\alpha\sinh\alpha b - 2\delta_2\beta\sin\beta b - (\delta_1 + \delta_2)\{\gamma + (1/\gamma)(m\pi/a)^2\}\sinh\gamma b$$

$$a_{33} = \delta_2\beta[\{1 + \gamma^2(a/m\pi)^2\}\cosh\gamma b - 2\cos\beta b]$$

The stability determinant $|a_{ij}| = 0$ yields values of $k$ for given values of $a/b$, $S$, $m$, and $\nu$; the minimum $k$ values for $\nu = 0.3$ are shown in Fig. 2. The mode shape is then determined by using Eqs. (24) and (25).

**Plate Simply Supported at $y = 0$ and $y = b$**

Applying the boundary conditions

$$W(0) = \Psi_x(0) = \Psi_y'(0) = W(b) = \Psi_x(b) = \Psi_y'(b) = 0$$

leads to the conclusions that $A = B = D = E = F = 0$ and that $\sin\beta b = 0$. Therefore $\beta b = \pi$ is the desired root of the stability determinant and a closed form expression for $k$ is given by

$$k = (mb/a + a/mb)^2/\{1 + S[(mb/a)^2 + 1]\} \tag{26}$$

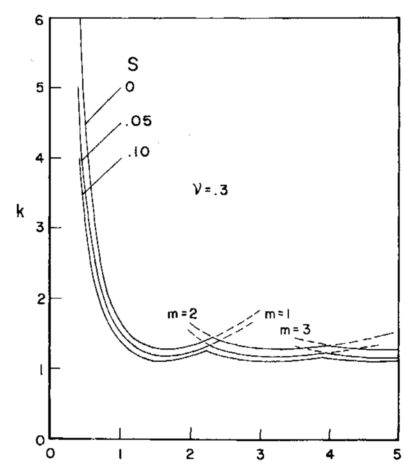


**Fig. 2 Buckling coefficient for $x$ sides simply supported, one side free and one side clamped.**

** A better form for computation is obtained by dividing the $a_{ij}$'s by the factor $\cosh\alpha b \cosh\gamma b$.

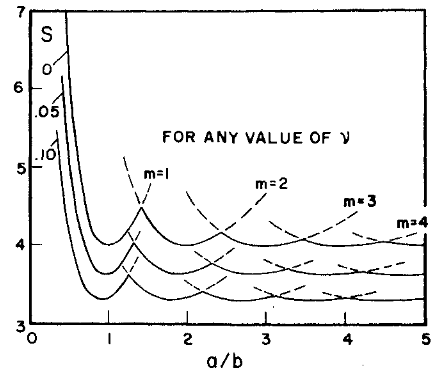


**Fig. 3 Buckling coefficient for all sides simply supported.**

The minimum values of $k$ are shown in Fig. 3 and the mode shape is given by

$$W(y) = C\sin\beta y \tag{27}$$

**Plate Clamped at $y = 0$ and $y = b$**

Applying the boundary conditions

$$W(0) = \Psi_x(0) = \Psi_y(0) = W(b) = \Psi_x(b) = \Psi_y(b) = 0$$

leads to the conclusions that

$$D = -B, \quad E = \gamma(a/m\pi)[-\delta_1\alpha A + \delta_2\beta C],$$
$$F = -(m\pi/a)(\delta_1 + \delta_2)B \tag{28}$$

and that

$$[a_{ij}]\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\} \tag{29}$$

where††

$$a_{11} = \sinh\alpha b, \quad a_{12} = \cosh\alpha b - \cos\beta b, \quad a_{13} = \sin\beta b$$

$$a_{21} = [(m\pi/a)\sinh\alpha b - \alpha\gamma(a/m\pi)\sinh\gamma b]\delta_1$$

$$a_{22} = [\delta_1\cosh\alpha b + \delta_2\cos\beta b - (\delta_1 + \delta_2)\cosh\gamma b](m\pi/a)$$

$$a_{23} = [\beta\gamma(a/m\pi)\sinh\gamma b - (m\pi/a)\sin\beta b]\delta_2$$

$$a_{31} = [\cosh\alpha b - \cosh\gamma b]\delta_1\alpha$$

$$a_{32} = \delta_1\alpha\sinh\alpha b - \delta_2\beta\sin\beta b - (1/\gamma)(m\pi/a)^2(\delta_1 + \delta_2)\sinh\gamma b$$

$$a_{33} = [\cosh\gamma b - \cos\beta b]\delta_2\beta$$

The stability determinant $|a_{ij}| = 0$ yields values of $k$ for given values of $a/b$, $S$, $m$, and $\nu$; the minimum $k$ values for $\nu = 0.3$ are shown in Fig. 4. The mode shape is then determined by using Eqs. (28) and (29).

## Discussion and Summary

Figures 1–4 show two general trends. Firstly, for $a/b < 0.6$, particularly large decreases of $k$ can occur for a given change in $S$. Secondly it is clear that increasing boundary restraint makes the buckling load more dependent on $S$, the effect being deleterious. A summary graph, Fig. 5, plots the minimum buckling coefficient $k_{MIN}$ vs $S$ for the four problems considered. It is seen that d $k_{MIN}/dS$ increases negatively as boundary restraint increases, for any $S$ = const value. Extrapolating the general trends of Figs. 1–5, it is to be expected that plates with even more boundary restraint (for example a plate clamped on all four sides) will suffer more severe decreases in buckling load due to the added shear deformation effects as-

†† A better form for computation is obtained by dividing the $a_{ij}$'s by the factor $\cosh\alpha b \cosh\gamma b$.

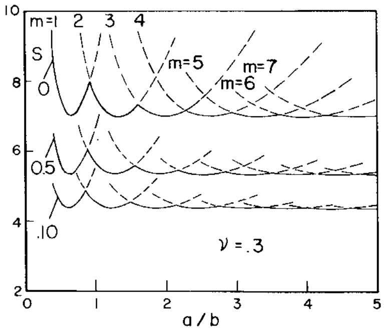


**Fig. 4 Buckling coefficient for $x$ sides simply supported, the other two sides clamped.**

sociated with transverse isotropy. Hence a useful extension of the present work would be the formulation of an approximate solution scheme for the plate stability equations (10–12). For example, a Galerkin technique using Timoshenko beam modes as the admissible functions might be a suitable and simple approximate solution scheme.

In summary, it has been shown that transverse isotropy, which accentuates the effects of shear deformation, lowers buckling loads for the cases considered; the decrease in the buckling loads becoming larger as 1) the boundary restraint is increased and 2) as the $a/b$ ratio becomes 0.6 or less. Hence, in particular, the engineer's conventional intuition as to the effects of boundary restraint in raising buckling loads must be carefully modified when dealing with transverely isotropic plates.

## Appendix A: Alternate Equations of Motion

Introducing the relations $\Phi = \partial\psi_x/\partial x + \partial\psi_y/\partial y$ and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, Eqs. (10–12) become

$$-Q(\Phi + \nabla^2 w) + S_x \frac{\partial^2(w + w_0)}{\partial x^2} + 2S_{xy}\frac{\partial^2(w + w_0)}{\partial x \partial y} + S_y \frac{\partial^2(w + w_0)}{\partial y^2} + M^*\ddot{w} = \frac{q}{\mathfrak{D}} \quad \text{(A1)}$$

$$\tfrac{1}{2}[(1 - \nu)\nabla^2\psi_x + (1 + \nu)\partial\Phi/\partial x] - Q(\psi_x + \partial w/\partial x) - \ddot{\psi}_x M = 0 \quad \text{(A2)}$$

$$\tfrac{1}{2}[(1 - \nu)\nabla^2\psi_y + (1 + \nu)\partial\Phi/\partial y] - Q(\psi_y + \partial w/\partial y) - \ddot{\psi}_y M = 0 \quad \text{(A3)}$$

Differentiate (A2) with respect to $x$ and (A3) with respect to $y$ and add the results to get

$$\nabla^2\Phi - M\ddot{\Phi} - Q\Phi = Q\nabla^2 w \quad \text{(A4)}$$

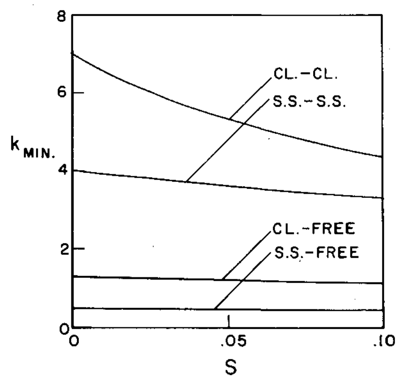


**Fig. 5 Minimum buckling coefficient vs $S$ for $x$ sides simply supported, the $y$ sides supported as indicated.**

Solve (A1) for $\Phi$ to yield

$$\Phi = \frac{1}{Q}\left[-Q\nabla^2 w + S_x\frac{\partial^2(w + w_0)}{\partial x^2} + 2S_{xy}\frac{\partial^2(w + w_0)}{\partial x \partial y} + S_y\frac{\partial^2(w + w_0)}{\partial y^2} + M^*\ddot{w} - \frac{q}{\mathfrak{D}}\right] \quad \text{(A5)}$$

and inserting (A5) into (A4) yields an equation for just $w(x,y,t)$.

$$\nabla^4 w - (M + M^*)\nabla^2\ddot{w} + M^*(\ddddot{w} + Q\ddot{w}) + S_x\left[Q\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2(\nabla^2 w)}{\partial x^2} + M\frac{\partial^2\ddot{w}}{\partial x^2}\right] + 2S_{xy}\left[Q\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial^2(\nabla^2 w)}{\partial x \partial y} + M\frac{\partial^2\ddot{w}}{\partial x\partial y}\right] + S_y\left[Q\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2(\nabla^2 w)}{\partial y^2} + M\frac{\partial^2\ddot{w}}{\partial y^2}\right] = \frac{1}{\mathfrak{D}}[Qq - \nabla^2 q + \ddot{q}] + S_x\left[\frac{\partial^2(\nabla^2 w_0)}{\partial x^2} - Q\frac{\partial^2 w_0}{\partial x^2}\right] + 2S_{xy}\left[\frac{\partial^2(\nabla^2 w_0)}{\partial x\partial y} - Q\frac{\partial^2 w_0}{\partial x\partial y}\right] + S_y\left[\frac{\partial^2(\nabla^2 w_0)}{\partial y^2} - Q\frac{\partial^2 w_0}{\partial y^2}\right] \quad \text{(A6)}$$

## Appendix B: A Summary of Some Simple $y = \tilde{y}$ Boundary Conditions

In what follows, the physical boundary conditions are given on the left and the corresponding mathematical boundary conditions are given on the right.

**Simply Supported at $y = \tilde{y}$**

$$\left.\begin{aligned} w(x,\tilde{y}) &= 0 \\ \psi_x(x,\tilde{y}) &= 0 \\ M_y(x,\tilde{y}) &= 0 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} W(\tilde{y}) &= 0 \\ \Psi_x(\tilde{y}) &= 0 \\ \Psi_y'(\tilde{y}) &= 0 \end{aligned}\right. \quad \text{(B1)}$$

**Clamped at $y = \tilde{y}$**

$$\left.\begin{aligned} w(x,\tilde{y}) &= 0 \\ \psi_x(x,\tilde{y}) &= 0 \\ \psi_y(x,\tilde{y}) &= 0 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} W(\tilde{y}) &= 0 \\ \Psi_x(\tilde{y}) &= 0 \\ \Psi_y(\tilde{y}) &= 0 \end{aligned}\right. \quad \text{(B2)}$$

**Free at $y = \tilde{y}$**

$$\left.\begin{aligned} M_y(x,\tilde{y}) &= 0 \\ M_{yx}(x,\tilde{y}) &= 0 \\ Q_y(x,\tilde{y}) &= 0 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} \nu(m\pi/a)\Psi_x(\tilde{y}) - \Psi_y'(\tilde{y}) &= 0 \\ \Psi_x'(\tilde{y}) + (m\pi/a)\Psi_y(\tilde{y}) &= 0 \\ W'(\tilde{y}) + \Psi_y(\tilde{y}) &= 0 \end{aligned}\right. \quad \text{(B3)}$$

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